

On metric of quantum channel spaces

Keiji Matsumoto

National Institute of Informatics, Tokyo, Japan

June 3, 2010

Abstract

So far, there have been plenty of literatures on the metric in the space of probability distributions and quantum states. As for channels, however, only a little had been known. In this paper, we impose monotonicity by concatenation of channels before and after the given channel families, and invariance by tensoring identity channels. Under these axioms, we identify the largest and the smallest metrics. Also, we studied asymptotic theory of metric in parallel and adaptive repetition settings, and applied them to the study of channel estimation. First we express the achievable lower bound of the mean square error (MSE) of an estimate by a monotone channel metric, and show this equals $O(1/n)$ for noisy channels, where n is the number of times of channel use. This result shows Heisenberg rate, or $O(1/n^2)$ -rate of the MSE observed in case of estimation of unitary, collapses with very small arbitrary noise.

1 Introduction

The aim of the manuscript is to characterize monotone (not necessarily Riemannian) metric in the space of quantum channels, or CPTP maps, and application of the theory to channel estimation problem.

So far, there have been plenty of literatures on the metric in the space of probability distributions and quantum states. Cencov, sometime in 1970s, proved the monotone metric in probability distribution space is unique up to constant multiple, and identical to Fisher information metric [4]. He also discussed invariant connections in the same space. Amari and others independently worked on the same objects, especially from differential geometrical view points, and applied to number of problems in mathematical statistics, learning theory, time series analysis, dynamic systems, control theory, and so on [1][2]. Quantum mechanical states are discussed in literatures such as [2][5][5][11]. Among them Petz [11] characterized all the monotone metrics in the quantum state space using operator mean.

As for channels, however, only a little had been known. To my knowledge, there had been no study about axiomatic characterization of distance measures in the classical or quantum channel space.

In this paper, we impose monotonicity by concatenation of channels before and after the given channel families, and invariance by tensoring identity channels. Notably, we do *not* suppose a metric is Riemannian, since, as was shown in , this assumption is not compatible with other assumptions.

Under these axioms, we identify the largest and the smallest metrics. Also, we studied asymptotic theory of metric in parallel and adaptive repetition settings, and applied them to the study of channel estimation. First we express the achievable lower bound of the mean square error (MSE) of an estimate by a monotone channel metric, and show this equals $O(1/n)$ for noisy channels, where n is the number of times of channel use. This result shows Heisenberg rate, or $O(1/n^2)$ -rate of the MSE observed in case of estimation of unitary, collapses with very small arbitrary noise.

2 Notations and conventions

- $\mathcal{H}_{\text{in}} (\mathcal{H}_{\text{out}})$: the Hilbert space for the input (output)
- $\mathcal{S}_{\text{in}} (\mathcal{S}_{\text{out}})$: the totality of the quantum states living in $\mathcal{H}_{\text{in}} (\mathcal{H}_{\text{out}})$. In this paper, the existence of density operator is always assumed. Hence, $\mathcal{S}_{\text{in}} (\mathcal{S}_{\text{out}})$ is equivalent to the totality of density operators.
- $\mathcal{S}(\mathcal{H})$: the totality of the quantum states living in \mathcal{H} .
- \mathcal{QC} : the totality of channels which sends an element of \mathcal{S}_{in} to an element of \mathcal{S}_{out}
- $\mathcal{QC}(\mathcal{S}_1, \mathcal{S}_2)$: the totality of channels which sends an element of \mathcal{S}_1 to an element of \mathcal{S}_2 . Abbreviated form \mathcal{QC} indicates that $(\mathcal{S}_1, \mathcal{S}_2) = (\mathcal{S}_{\text{in}}, \mathcal{S}_{\text{out}})$. Also, $\mathcal{QC}(\mathcal{S}(\mathcal{H}_1), \mathcal{S}(\mathcal{H}_2))$ is abbreviated as $\mathcal{QC}(\mathcal{H}_1, \mathcal{H}_2)$. Also, $\mathcal{QC}(\mathcal{S})$ and $\mathcal{QC}(\mathcal{H})$ means $\mathcal{QC}(\mathcal{S}, \mathcal{S})$ and $\mathcal{QC}(\mathcal{H}, \mathcal{H})$, respectively.
- A quantum state ρ is identified with the channel which sends all the input states to ρ .
- $\mathcal{T}(\cdot)$: tangent space
- δ etc. : an element of $\mathcal{T}_\rho(\mathcal{S}_{\text{in}})$, $\mathcal{T}_p(\mathcal{P}_\Omega)$, etc.
- Δ etc. : an element of $\mathcal{T}_\Phi(\mathcal{QC})$
- An element δ of $\mathcal{T}_\rho(\mathcal{S})$ etc. is identified with an element of $\tau_c(\mathcal{S}_{\text{in}})$ such that $\text{tr}\delta = 0$.
- $g_\rho(\delta)$: square of a norm in $\mathcal{T}_\rho(\mathcal{S})$
- $h_p(\delta)$: square of a norm in $\mathcal{T}_p(\mathcal{P})$
- $G_\Phi(\Delta)$: square of a norm in $\mathcal{T}_\Phi(\mathcal{QC})$
- $J_p(\delta)$: classical Fisher information

- $J_p^S(\delta)$: SLD Fisher information. $J_p^S(\delta) := \text{tr} \rho (L_\rho^S)^2$, where L_ρ^S is symmetric logarithmic derivative (SLD), or
the solution to the equation $\delta = \frac{1}{2} (L_\rho^S \rho + \rho L_\rho^S)$.
- $J_p^R(\delta)$: RLD Fisher information. $J_p^R(\delta) := \Re \text{tr} \rho (L_\rho^R)^\dagger L_\rho^R$, where L_ρ^R is right logarithmic derivative (RLD), or
the solution to the equation $\delta = L_\rho^R \rho$.
- The local data at p , etc.: the pair $\{p, \delta\}$, etc.
- $\Phi(\cdot|x) \in \mathcal{P}_{\text{out}}$: the distribution of the output alphabet when the input is x
- $\Delta(\cdot|x) \in \mathcal{T}_p(\mathcal{P}_{\text{out}})$ is defined as the infinitesimal increment of above
- **I**: identity
- $\delta_e := \frac{\delta}{p}$
- $\delta^{(n)} := \delta \otimes p^{\otimes n-1} + p \otimes \delta \otimes p^{\otimes n-2} + \dots + p^{\otimes n-1} \otimes \delta \in \mathcal{T}_p(\mathcal{P}^{\otimes n})$
- $\delta_e^{(n)} := \delta_e \otimes 1^{\otimes n-1} + 1 \otimes \delta_e \otimes 1^{\otimes n-2} + \dots + 1^{\otimes n-1} \otimes \delta_e$
- $\Delta^{(n)} := \Delta \otimes \Phi^{\otimes n-1} + \Phi \otimes \Delta \otimes \Phi^{\otimes n-2} + \dots + \Phi^{\otimes n-1} \otimes \Delta \in \mathcal{T}_\Phi(\mathcal{C}^{\otimes n})$
- $\mathcal{N}(a, \sigma^2)$: Gaussian with mean a and the variance σ^2 .
- $\delta \mathcal{N}(a, \sigma^2)$ is singed measure defined by $\delta \mathcal{N}(a, \sigma^2)(B) = \frac{1}{\sqrt{2\pi}\sigma} \int_B \frac{x-a}{\sigma^2} \exp\left[-\frac{1}{2\sigma^2}(x-a)^2\right] dx$. This corresponds to the tangent vector of Gaussian shift family with the variance σ^2 .

3 Single-copy theory

3.1 Axioms

$$\text{(M)} \quad G_\Phi(\Delta) \geq G_{\Phi \circ \Psi}(\Delta \circ \Psi), \quad G_\Phi(\Delta) \geq G_{\Psi \circ \Phi}(\Psi \circ \Delta)$$

$$\text{(E)} \quad G_{\Phi \otimes \mathbf{I}}(\Delta \otimes \mathbf{I}) = G_\Phi(\Delta)$$

$$\text{(N)} \quad G_p(\delta) = J_p(\delta)$$

3.2 Estimation of channel and G^{\min}

Consider estimation of an unknown channel, which is drawn from the family $\{\Phi_\theta\}_{\theta \in \mathbb{R}}$, where $\theta \in \mathbb{R}$ is unknown scalar parameter. The asymptotic mean-square error of probability distribution and quantum state is inversely proportional to Fisher information $J_p(\delta)$ and SLD Fisher information $J_\rho^S(\delta)$, respectively. Hence, it is natural to consider

$$G_\Phi^{\min}(\Delta) := \sup_{\substack{\rho \in \mathcal{S}(\mathcal{H}_n \otimes \mathcal{K}) \\ M \in \mathcal{M}_{\text{out}}}} J_{M \circ (\Phi \otimes \mathbf{I})(\rho)}(M \circ (\Delta \otimes \mathbf{I})(\rho)) = \sup_{\rho \in \mathcal{S}(\mathcal{H}_n \otimes \mathcal{K})} J_{(\Phi \otimes \mathbf{I})(\rho)}^S((\Delta \otimes \mathbf{I})(\rho)),$$

where the identity is due to characterization of SLD Fisher information in [10]:

Theorem 1 [10]

$$J_\rho^S(\delta) = \sup_M J_{M(\rho)}(M(\delta)).$$

Theorem 2 Suppose (M) and (N) hold. Then,

$$G_\Phi(\Delta) \geq G_\Phi^{\min}(\Delta)$$

Also, $G_\Phi^{\min}(\Delta)$ satisfies (M), (E), and (N).

Proof.

$$G_\Phi(\Delta) = G_{\Phi \otimes \mathbf{I}}(\Delta \otimes \mathbf{I}) \geq G_{M \circ (\Phi \otimes \mathbf{I})(\rho)}(M \circ (\Delta \otimes \mathbf{I})(\rho)) = J_{M \circ (\Phi \otimes \mathbf{I})(\rho)}(M \circ (\Delta \otimes \mathbf{I})(\rho)).$$

Hence, we have inequality. That $G_\Phi^{\min}(\Delta)$ satisfies (M1), (M2), (E), and (N) is trivial. ■

3.3 Tangent simulation of channel family and G^{\max}

Suppose we have to fabricate a channel Φ_θ , which is drawn from a family $\{\Phi_\theta\}$, without knowing the value of θ but with a probability distribution q_θ or ρ_θ drawn from a family $\{q_\theta\}$ or $\{\rho_\theta\}$. More specifically, we need a channel Λ with

$$\Phi_\theta = \Lambda \circ (\mathbf{I} \otimes q_\theta), \tag{1}$$

or

$$\Phi_\theta = \Lambda \circ (\mathbf{I} \otimes \sigma_\theta), \tag{2}$$

Here, note that Λ should not vary with the parameter θ . Note also that the former is a special case of the latter. Also, giving the value of θ with infinite precision corresponds to the case of having the delta distribution peaked at θ . This is channel version of randomization criteria for deficiency, which is a fundamental concept in statistical decision theory [12].

Differentiating the both ends of (1), (??), and (2), we obtain

$$\Delta = \Lambda \circ (\mathbf{I} \otimes \delta), \quad (3)$$

where $\Delta \in \mathcal{T}_\Phi(\mathcal{C})$, $\delta \in \mathcal{T}_q(\mathcal{P}_{\text{pr}})$ (, or $\mathcal{T}_\rho(\mathcal{S}_{\text{pr}})$).

In the manuscript, we consider *classical tangent simulation* (, or *quantum tangent simulation*), or the triplet $\{q, \delta, \Lambda\}$ (, or $\{\sigma, \delta, \Lambda\}$) satisfying (1) (, or (2)) and (3), at the point $\Phi_\theta = \Phi$ only. Note that classical and quantum tangent simulation of $\{\Phi, \Delta\}$ is equivalent to simulation of the channel family $\{\Phi_{\theta+t} = \Phi + t\Delta\}_t$. (This is channel analogue of local deficiency in statistical decision theory [12].)

Based on tangent simulation, we define :

$$\begin{aligned} G_\Phi^{\max}(\Delta) &:= \inf \{J_q(\delta) ; \{\Lambda, q, \delta\} \text{ is a classical tangent simulation of } \{q, \delta\} \}, \\ &= \inf \{J_\sigma^R(\delta) ; \{\Lambda, \sigma, \delta\} \text{ is a quantum tangent simulation of } \{q, \delta\} \}, \end{aligned}$$

where the identity in the second line is due to characterization of RLD in [6].

Theorem 3 *Suppose (M), (E) and (N) hold. Then*

$$G_\Phi(\Delta) \leq G_\Phi^{\max}(\Delta).$$

Also, $G_\Phi^{\max}(\Delta)$ satisfies (M), (E), and (N).

Proof.

$$J_q(\delta) = G_q(\delta) = G_{\mathbf{I} \otimes q}(\mathbf{I} \otimes \delta) \geq G_{\Lambda \circ (\mathbf{I} \otimes q)}(\Lambda \circ (\mathbf{I} \otimes \delta)) = G_\Phi(\Delta).$$

So we have the inequality. That $G_\Phi^{\max}(\Delta)$ satisfies (M), (E), and (N) is trivial. ■

Corollary 4

$$G_\Phi^{\max}(\Delta) \geq G_\Phi^{\min}(\Delta).$$

Example 5 [3] *Consider the following family of channels :*

$$\Lambda_\theta(\rho) = (1 - p_x - p_y - p_z)\rho + p_x X \rho X + p_y Y \rho Y + p_z Z \rho Z,$$

where X, Y, Z are Pauli matrices and p_x, p_y, p_z are scalar functions of θ . In other words, consider random application of Pauli matrices with unknown probability distribution $p_\theta = (p_x(\theta), p_y(\theta), p_z(\theta))$. Therefore,

$$G_{\Lambda_\theta}^{\max}(\Delta_\theta) \leq J_{p_\theta}(\delta_\theta)$$

where $\Delta_\theta = d\Lambda_\theta/d\theta$ and $\delta_\theta = dp_\theta/d\theta$. On the other hand, let

$$\begin{aligned} |\text{Bell}_1\rangle &:= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\ |\text{Bell}_2\rangle &:= I \otimes X |\text{Bell}_1\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\ |\text{Bell}_3\rangle &:= \sqrt{-1} (I \otimes Y) |\text{Bell}_1\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \\ |\text{Bell}_4\rangle &:= (I \otimes Z) |\text{Bell}_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle). \end{aligned}$$

Observe that they are orthogonal with each other. Hence, as Fujiwara and others had pointed out, by inserting one part of $|\text{Bell}_1\rangle$ and measuring the output, we can identify which Pauli matrix was multiplied. Therefore,

$$G_{\Lambda_\theta}^{\min}(\Delta_\theta) \geq J_{p_\theta}(\delta_\theta).$$

Hence, after all,

$$G_{\Lambda_\theta}^{\min}(\Delta_\theta) = G_{\Lambda_\theta}^{\max}(\Delta_\theta) = g_{p_\theta}(\delta_\theta).$$

3.4 Quantum states

A quantum state can be viewed as a quantum channel with constant output. In [6], (M2) and (N) implies that

$$J_\rho^S(\delta) \leq G_\rho(\delta) \leq J_\rho^R(\delta).$$

4 Asymptotic theory : parallel version

Parallel use of n of Φ means that we are given $\Phi^{\otimes n}$, send in a big input $\rho \in \mathcal{S}_{\text{in}}^n$ to $\Phi^{\otimes n}$.

4.1 Additional axioms: parallel version

(Ap) (parallel asymptotic weak additivity) $\lim_{n \rightarrow \infty} \frac{1}{n} G_{\Phi^{\otimes n}}(\Delta^{(n)}) = G_\Phi(\Delta)$

4.2 $G^{p,\min}$ and $G^{p,\max}$

We define

$$\begin{aligned} G_\Phi^{\min,p}(\Delta) &:= \lim_{n \rightarrow \infty} \frac{1}{n} G_{\Phi^{\otimes n}}^{\min}(\Delta^{(n)}), \\ G_\Phi^{\max,p}(\Delta) &:= \lim_{n \rightarrow \infty} \frac{1}{n} G_{\Phi^{\otimes n}}^{\max}(\Delta^{(n)}), \end{aligned}$$

Theorem 6 (M) , (E) , (Ap) , and (N) implies that

$$G_{\Phi}^{\min,p}(\Delta) \leq G_{\Phi}(\Delta) \leq G_{\Phi}^{\max,p}(\Delta).$$

Also, $G_{\Phi}^{\min,p}(\Delta)$ and $G_{\Phi}^{\max,p}(\Delta)$ satisfy (M) , (E) , (Ap) , and (N) .

Proof. By Theorem 2 and Theorem 3,

$$\frac{1}{n} G_{\Phi^{\otimes n}}^{\min}(\Delta^{(n)}) \leq \frac{1}{n} G_{\Phi^{\otimes n}}(\Delta^{(n)}) \leq \frac{1}{n} G_{\Phi^{\otimes n}}^{\max}(\Delta^{(n)}).$$

Taking sup of the last end and letting $n \rightarrow \infty$, we have the assertion That $G_{\Phi}^{\min,p}(\Delta)$ and $G_{\Phi}^{\max,p}(\Delta)$ satisfy (M) , (E) , (Ap) , and (N) is trivial. ■

5 Asymptotic theory : adaptive version

5.1 Adaptive repetition

In estimating channel, we may use it sequentially, applying some channel Ψ_{κ} between k th and $(k-1)$ th application of $\Phi \otimes \mathbf{I}$:

$$\prod_{k=n}^1 \{(\Phi \otimes \mathbf{I}) \circ \Psi_{\kappa}\}. \quad (4)$$

To indicate such use, we define n -adaptive repetition of Φ by $\Phi^{\#n}$. Formal definition is that $\Phi^{\#n}$ is a linear map which sends the pair $\Psi^n := (\Psi_1, \Psi_2, \dots, \Psi_n)$ to (4). We also define

$$\Delta^{(\#n)} := \Delta \# \Phi^{\#n-1} + \Phi \# \Delta \# \Phi^{\#n-2} + \dots + \Phi^{\#n-1} \# \Delta.$$

Here, Δ is identified with a linear map from operators to operators. For the sake of briefness, we denote:

$$\begin{aligned} \Phi^{\#n}(\Psi^n) &:= \prod_{k=n}^1 \{(\Phi \otimes \mathbf{I}) \circ \Psi_{\kappa}\}, \\ \Delta^{(\#n)}(\Psi^n) &:= \{(\Delta \otimes \mathbf{I}) \circ \Psi_n\} \circ \prod_{k=n-1}^1 \{(\Phi \otimes \mathbf{I}) \circ \Psi_{\kappa}\} \\ &\quad + \{(\Phi \otimes \mathbf{I}) \circ \Psi_n\} \circ \{(\Delta \otimes \mathbf{I}) \circ \Psi_{n-1}\} \circ \prod_{k=n-2}^1 \{(\Phi \otimes \mathbf{I}) \circ \Psi_{\kappa}\} + \dots \\ &\quad + \prod_{k=n}^2 \{(\Phi \otimes \mathbf{I}) \circ \Psi_{\kappa}\} \circ \{(\Delta \otimes \mathbf{I}) \circ \Psi_1\}. \end{aligned} \quad (5)$$

One can define

$$G_{\Delta^{\#n}}^{\min}(\Phi^{\#n}) := \sup \left\{ J_{\tilde{p}}(\tilde{\delta}) ; \tilde{p} = M \circ \Phi^{\#n}(\Psi^n)(\rho), \tilde{\delta} = M \circ \Delta^{(\#n)}(\Psi^n)(\rho) \right\},$$

$$G_{\Delta^{\#n}}^{\max}(\Phi^{\#n}) := \inf J_q(\delta'),$$

where the infimum in the second definition is taken over all $\{q, \delta'\}$ which satisfies for some $\Lambda^n := (\Lambda_1, \Lambda_2, \dots, \Lambda_n)$

$$\prod_{k=n}^1 \{(\Lambda_k \otimes \mathbf{I} \otimes \mathbf{I}) \circ \Psi_k \otimes \mathbf{I}\}(\rho \otimes q) = \Phi^{\#n}(\Psi^n)(\rho) \quad (6)$$

$$\prod_{k=n}^1 \{(\Lambda_k \otimes \mathbf{I} \otimes \mathbf{I}) \circ (\Psi_k \otimes \mathbf{I})\}(\rho \otimes \delta) = \Delta^{(\#n)}(\Psi^n)(\rho). \quad (7)$$

Classical tangent simulation of $\{\Phi^{\#n}, \Delta^{(\#n)}\}$ is defined as a pair $\{\Lambda^n, q, \delta'\}$ with (6), (7).

5.2 $G^{\min,a}$ and $G^{\max,a}$

We define

$$G_{\Phi}^{\min,a}(\Delta) := \lim_{n \rightarrow \infty} \frac{1}{n} G_{\Phi^{\#n}}^{\min}(\Delta^{(\#n)}),$$

$$G_{\Phi}^{\max,a}(\Delta) := \lim_{n \rightarrow \infty} \frac{1}{n} G_{\Phi^{\#n}}^{\max}(\Delta^{(\#n)}).$$

Then we have the following theorems.

Theorem 7

$$G_{\Phi}^{\min,a}(\Delta) \leq G_{\Phi}^{\max,a}(\Delta).$$

Also, $G_{\Phi}^{\min,a}(\Delta)$ and $G_{\Phi}^{\max,a}(\Delta)$ satisfy (M), (E) and (N) and (Aa).

Proof. That $G_{\Phi}^{\min,a}(\Delta)$ and $G_{\Phi}^{\max,a}(\Delta)$ satisfy (M), (E), and (N) is trivial. So we prove the inequality. Let

$$\tilde{p}^n = M \circ \Phi^{\#n}(\Psi^n)(\rho^n), \tilde{\delta}^n = M \circ \Delta^{(\#n)}(\Psi^n)(\rho^n)$$

and let $\{\Lambda^n, q^n, \delta'^n\}$ be a tangent simulation of $\{\Phi^{\#n}, \Delta^{(\#n)}\}$. Then by monotonicity of Fisher information, we have

$$J_{q^n}(\delta'^n) \geq J_{\tilde{p}^n}(\tilde{\delta}^n).$$

Hence, taking infimum of the LHS and the maximum of the RHS and letting $n \rightarrow \infty$, we obtain the second inequality. ■

Proposition 8

$$G_{\Phi}^{\min}(\Delta) \leq G_{\Phi}^{\min,p}(\Delta) \leq G_{\Phi}^{\min,a}(\Delta)$$

$$\leq G_{\Phi}^{\max,p}(\Delta) \leq G_{\Phi}^{\max,a}(\Delta) \leq G_{\Phi}^{\max}(\Delta).$$

Proof. Non-trivial part is $G_{\Phi}^{\min,a}(\Delta) \leq G_{\Phi}^{\max,p}(\Delta)$. To prove this, it suffices to show $G_{\Phi}^{\min,a}(\Delta)$ satisfies (A1). Consider

$$\begin{aligned} \frac{1}{m} G_{\Phi^{\otimes m}}^{\min,a}(\Delta^{(m)}) &= \lim_{n \rightarrow \infty} \frac{1}{nm} G_{(\Phi^{\otimes m})^{\#n}}^{\min} \left(\left(\Delta^{(m)} \right)^{\#n} \right) \\ &\leq \lim_{n \rightarrow p} \frac{1}{nm} G_{\Phi^{\#nm}}^{\min}(\Delta^{(\#nm)}) \\ &= G_{\Phi}^{\min,a}(\Delta). \end{aligned}$$

Here, the inequality in the second line holds since operations allowed in the optimization problem used to define the RHS quantity is richer than those used to define the LHS quantity.

On the other hand, $(\Phi^{\otimes m})^{\#n}$ can be thought as m -parallelization of n -adaptive sequence. If we restrict Ψ^n , M , and ρ so that there is no entanglement nor interaction between these parallelization, obtained Fisher information becomes in general smaller. Hence,

$$\begin{aligned} \frac{1}{nm} G_{(\Phi^{\otimes m})^{\#n}}^{\min} \left(\left(\Delta^{(m)} \right)^{\#n} \right) &= \frac{1}{nm} \sup \left\{ J_{\tilde{p}}(\tilde{\delta}) ; \tilde{p} = M \circ (\Phi^{\otimes m})^{\#n}(\Psi^n)(\rho), \tilde{\delta} = M \circ \left(\Delta^{(m)} \right)^{\#n}(\Psi^n)(\rho) \right\} \\ &\geq \frac{1}{nm} \sup \left\{ J_{\tilde{p}}(\tilde{\delta}) ; \tilde{p} = \left(\tilde{M} \circ (\Phi^{\#n}) \left(\tilde{\Psi}^n \right) (\tilde{\rho}) \right)^{\otimes m}, \tilde{\delta} = \left(M \circ \Delta^{(\#n)} \left(\tilde{\Psi}^n \right) (\tilde{\rho}) \right)^{(\#m)} \right\} \\ &= \frac{1}{nm} \sup \left\{ J_{\tilde{p}^{\otimes m}}(\tilde{\delta}^{\otimes m}) ; \tilde{p} = \tilde{M} \circ (\Phi^{\#n}) \left(\tilde{\Psi}^n \right) (\tilde{\rho}), \tilde{\delta} = M \circ \Delta^{(\#n)} \left(\tilde{\Psi}^n \right) (\tilde{\rho}) \right\} \\ &= \frac{1}{n} \sup \left\{ J_{\tilde{p}}(\tilde{\delta}) ; \tilde{p} = \tilde{M} \circ (\Phi^{\#n}) \left(\tilde{\Psi}^n \right) (\tilde{\rho}), \tilde{\delta} = M \circ \Delta^{(\#n)} \left(\tilde{\Psi}^n \right) (\tilde{\rho}) \right\} \\ &= \frac{1}{n} G_{\Phi^{\#n}}^{\min}(\Delta^{(\#n)}). \end{aligned}$$

By $n \rightarrow \infty$, this yields

$$\frac{1}{m} G_{\Phi^{\otimes m}}^{\min,a}(\Delta^{(m)}) \geq G_{\Phi}^{\min,a}(\Delta).$$

After all, we have

$$\frac{1}{m} G_{\Phi^{\otimes m}}^{\min,a}(\Delta^{(m)}) = G_{\Phi}^{\min,a}(\Delta),$$

and our assertion is proved. ■

Conjecture 9 $G_{\Phi}^{\max,a}(\Delta) = G_{\Phi}^{\max}(\Delta)$.

5.3 Examples

5.3.1 Unital qubit channels

In this case, since $G_{\Phi}^{\max}(\Delta) = G_{\Phi}^{\min}(\Delta)$, it follows that

$$G_{\Phi}^{\min}(\Delta) = G_{\Phi}^{\min,p}(\Delta) = G_{\Phi}^{\min,a}(\Delta) = G_{\Phi}^{\max,p}(\Delta) = G_{\Phi}^{\max,a}(\Delta) = G_{\Phi}^{\max}(\Delta).$$

5.3.2 QC channels

If $\{\Phi_\theta\}$ is a QC channel, $G_\Phi^{\min}(\Delta) = G_\Phi^{\min,p}(\Delta) = G_\Phi^{\min,a}(\Delta)$. This is proved as follows. If only classical data is fed to the succeeding measurement, the fisher information obtained is $G_\Phi^{\min}(\Delta)$, due to. In general, we may have large input state $\rho_{\text{in}} \in \mathcal{S}(\mathcal{H}_{\text{in}} \otimes \mathcal{K})$, where the measurement is applied only to \mathcal{H}_{in} , and we are left with the measurement result (classical information) and the post-measurement state in \mathcal{K} . This post-measurement state is determined by the measurement data, and therefore not needed given the measurement result. Since it can be fabricated whenever necessary.

5.3.3 Quantum states

A quantum state can be considered as a channel with constant output. Indeed, if g satisfies (M) and (N),

$$J_\rho^S(\delta) \leq g_\rho(\delta) \leq J_\rho^R(\delta).$$

[6]. Moreover, it is known that J^S and J^R satisfy not only (M) and (N), but also (A1).

5.3.4 Unitary channels and noisy channels

If $\{\Phi_\theta\}$ are unitary operations, $G_\Phi^{\min,p}(\Delta) = \infty$. Hence,

$$G_\Phi^{\min,p}(\Delta) = G_\Phi^{\min,a}(\Delta) = G_\Phi^{\max,p}(\Delta) = G_\Phi^{\max,a}(\Delta) = G_\Phi^{\max}(\Delta) = \infty. \quad (8)$$

If Φ is in the interior of \mathcal{QC} , and $\dim \mathcal{H}_{\text{in}} < \infty$, $\dim \mathcal{H}_{\text{out}} < \infty$, there is a $\varepsilon > 0$ such that

$$\Phi + \theta\Delta \in \mathcal{QC}, \quad \forall |\theta| \leq \varepsilon.$$

Then, $\{\Phi, \Delta\}$ can be simulated by probabilistic mixture of $\Phi + \varepsilon\Delta$ and $\Phi - \varepsilon\Delta$. More precisely, let $\Lambda \in \mathcal{QC}(\mathcal{H}_{\text{in}} \otimes \mathbb{C}^2, \mathcal{H}_{\text{out}})$ be a channel such that

$$\Lambda(\rho_{\text{in}} \otimes |0\rangle\langle 0|) = (\Phi + \varepsilon\Delta)(\rho_{\text{in}} \otimes |0\rangle\langle 0|),$$

$$\Lambda(\rho_{\text{in}} \otimes |1\rangle\langle 1|) = (\Phi - \varepsilon\Delta)(\rho_{\text{in}} \otimes |1\rangle\langle 1|),$$

q is the probability distribution on $\{0, 1\}$ with $q(0) = q(1) = \frac{1}{2}$, and $\delta(0) = (2\varepsilon)^{-1}$, $\delta(1) = -(2\varepsilon)^{-1}$.

Therefore,

$$G_\Phi^{\min,p}(\Delta) \leq G_\Phi^{\min,a}(\Delta) \leq G_\Phi^{\max,p}(\Delta) \leq G_\Phi^{\max,a}(\Delta) \leq G_\Phi^{\max}(\Delta) \leq (\varepsilon)^{-2} < \infty. \quad (9)$$

6 Asymptotic theory of estimation of noisy channels

6.1 Cramer-Rao type bound

An adaptive estimator of the channel family $\{\Phi_\theta\}$ is a sequence $\{\rho_{\text{in}}^n, \Psi^n, M^n\}_{n=1}^\infty$ of triplet of a pair of channels $\Psi^n := (\Psi_1, \Psi_2, \dots, \Psi_n)$, the input state $\rho^n \in \mathcal{S}(\mathcal{H}_{\text{in}} \otimes \mathcal{K}^n)$, and the measurement $M^n \in$

$\mathcal{S}(\mathcal{M}_{\text{out}} \otimes \mathcal{K}^n)$, which takes values in \mathbb{R} . $\{\rho_{\text{in}}^n, \Psi^n, M^n\}_{n=1}^\infty$ is said to be *asymptotically unbiased* if

$$\lim_{n \rightarrow \infty} \mathbb{E}_\theta [\{\rho_{\text{in}}^n, \Psi^n, M^n\}] = \theta, \quad \lim_{n \rightarrow \infty} \frac{d}{d\theta} \mathbb{E}_\theta [\{\rho_{\text{in}}^n, \Psi^n, M^n\}] = 1, \quad (10)$$

where $\mathbb{E}_\theta [\{\rho_{\text{in}}^n, \Psi^n, M^n\}]$ refers to the expectation of estimate obeying the probability distribution $M^n \circ \Phi^{\#n}(\Psi^n)(\rho_{\text{in}}^n)$. This is a regularity condition often imposed on estimators. Given an asymptotically unbiased estimator, one can define a measurement $M_{\theta_0}^n$ with measurement result

$$\begin{aligned} \check{\theta}_{\theta_0}^n &:= \frac{1}{\frac{d}{d\theta} b_{\theta_0}^n} (\hat{\theta}^n - b_{\theta_0}^n) + \theta_0, \\ \hat{\theta}^n &= \left(\frac{d}{d\theta} b_{\theta_0}^n \right) (\check{\theta}_{\theta_0}^n - \theta_0) + b_{\theta_0}^n \end{aligned}$$

where $\hat{\theta}^n$ is the measurement result of M^n and $b_{\theta_0}^n := \mathbb{E}_\theta [\{\rho_{\text{in}}^n, \Psi^n, M^n\}]$. Then, $M_{\theta_0}^n$ satisfies

$$\mathbb{E}_{\theta_0} [\{\rho_{\text{in}}^n, \Psi^n, M_{\theta_0}^n\}] = \theta_0, \quad \left. \frac{d}{d\theta} \mathbb{E}_\theta [\{\rho_{\text{in}}^n, \Psi^n, M_{\theta_0}^n\}] \right|_{\theta=\theta_0} = 1, \quad (11)$$

and θ

$$\begin{aligned} \liminf_{n \rightarrow \infty} n \mathbb{E}_{\theta_0} (\hat{\theta}^n - \theta_0)^2 &= \liminf_{n \rightarrow \infty} n \mathbb{E}_{\theta_0} \left(\left(\frac{d}{d\theta} b_{\theta_0}^n \right) (\check{\theta}_{\theta_0}^n - \theta_0) + b_{\theta_0}^n - \theta_0 \right)^2 \\ &= \liminf_{n \rightarrow \infty} n \mathbb{E}_{\theta_0} ((\check{\theta}_{\theta_0}^n - \theta_0) + b_{\theta_0}^n - \theta_0)^2 \\ &\geq \liminf_{n \rightarrow \infty} n \mathbb{E}_{\theta_0} (\check{\theta}_{\theta_0}^n - \theta_0)^2 \\ &\geq \left(\liminf_{n \rightarrow \infty} \frac{1}{n} J_{\tilde{p}^n}(\tilde{\delta}^n) \right)^{-1}, \end{aligned}$$

where $\tilde{p}^n := M \circ \Phi^{\#n}(\Psi^n)(\rho_{\text{in}}^n)$ and $\tilde{\delta}^n := M \circ \Delta^{(\#n)}(\Psi^n)(\rho_{\text{in}}^n)$.

Hence, we obtain the *Cramer-Rao type bound* [9]

$$\inf \left\{ \liminf_{n \rightarrow \infty} n \mathbb{E}_{\theta_0} (\hat{\theta}^n - \theta_0)^2 ; \{\rho_{\text{in}}^n, \mathfrak{F}^n, M^n\} \text{ with (10)} \right\} \geq \left(G_{\Phi}^{\min, a}(\Delta) \right)^{-1}. \quad (12)$$

Indeed, one can show the identity in (12) is achievable, if $G_{\Phi}^{\min, a}(\Delta) < \infty$ and some regularity conditions are satisfied [9].

6.2 On ‘Heisenberg rate’

If $\{\Phi_\theta\}$ are unitary operations, due to (8), (12) does not give any information on the efficiency. Indeed, a number of literatures show that $\mathbb{E}_{\theta_0} (\hat{\theta}^n - \theta_0) = O(\frac{1}{n^2})$ (Heisenberg rate), and some refers to application to metrology. However, the efficiency of the optimal estimator is very weak against the noise in the operations, as some authors have pointed out in some physical models.

Combination of (9) and (12) shows a general result [9] :

Theorem 10 Suppose $\dim \mathcal{H} < \infty$ and that there is a ε_θ such that $\Phi_\theta + \varepsilon_\theta (d\Phi_\theta/d\theta)$ and $\Phi - \varepsilon_\theta (d\Phi_\theta/d\theta)$ are completely positive. Then if $\{\rho_{\text{in}}^n, \mathfrak{F}^n, M^n\}$ satisfies (10),

$$\mathbb{E}_\theta (\hat{\theta}^n - \theta) = O\left(\frac{1}{n}\right), \forall \theta.$$

Therefore, whatever the noise it is, however small it is, Heisenberg rate collapses. Note Theorem can be easily extended to the case that θ is multi-dimensional. Obtaining estimate $\hat{\theta}^n$ of multi-dimensional parameter θ satisfying (10) for each components. Then, its first component $\hat{\theta}^{n,1}$ is a estimate of scalar parameter θ^1 with (10). Hence, due to Theorem 10, we have

$$\mathbb{E}_\theta \left\| \hat{\theta}^{n,1} - \theta^1 \right\|^2 \geq \mathbb{E}_\theta (\hat{\theta}^{n,1} - \theta^1) = O\left(\frac{1}{n}\right).$$

7 Asymptotic theory with approximation

7.1 Motivations

Axiom (N) is justified because this is consequence of the rest of the axioms and

$$(\mathbf{N}') \quad g_q(\delta') = 1, \text{ if } \{q, \delta'\} = \{N(0, 1), \delta N(0, 1)\}$$

(C1) (*parallel weak asymptotic continuity*) If $\|\Phi^n - \Phi^{\otimes n}\|_{\text{cb}} \rightarrow 0$ and $\frac{1}{\sqrt{n}} \|\Delta^n - \Delta^{(n)}\|_{\text{cb}} \rightarrow 0$ then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(G_{\Phi^n}(\Delta^n) - G_{\Phi^{\otimes n}}(\Delta^{(n)}) \right) \geq 0.$$

The proof uses asymptotic tangent simulation [8], which simulates $\{p^{\otimes n}, \delta^{(n)}\}$ by Gaussian shift $\{q, \delta'\} = \{N(0, 1), \delta N(0, 1)\}$ only approximately. Hence, for the sake of coherency, it is preferable to build a theory based on asymptotic tangent simulation.

7.2 Asymptotic tangent simulation (parallel) and $\tilde{G}^{p, \max}$

An *asymptotic parallel classical tangent simulation* is a sequence $\{q^n, \delta^n, \Lambda^n\}_{n=1}^\infty$ of $q^n \in \mathcal{P}_{\text{pr}}$ ($\sigma^n \in \mathcal{S}_{\text{pr}}$), $\delta^n \in \mathcal{T}_q(\mathcal{P}_{\text{pr}})$ and a CPTP map Λ^n , such that

$$\lim_{n \rightarrow \infty} \left\| \Phi^{\otimes n} - \Lambda^n(\mathbf{I} \otimes q^n) \right\|_{\text{cb}} = 0, \quad (13)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \Delta^{(n)} - \Lambda^n \circ (\mathbf{I} \otimes \delta^n) \right\|_{\text{cb}} = 0, \quad (14)$$

where $\Delta \in \mathcal{T}_\Phi(\mathcal{C})$, $\delta \in \mathcal{T}_q(\mathcal{P}_{\text{pr}})$.

Based on this, we define the following quantity.

$$\tilde{G}_{\Phi}^{p,\max}(\Delta) := \lim_{n \rightarrow \infty} \frac{1}{n} \inf \{J_{q^n}(\delta'^n); \{q^n, \delta'^n\} \text{ is a Gaussian shift with } (13), (14)\}.$$

(Here note that \lim always exists and finite.)

Also, we define

$$G_{\Phi}^{p,R}(\Delta) := \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{\rho} J_{\Phi^{\otimes n}(\rho)}^R(\Delta^{(n)}(\rho)).$$

Theorem 11 $(M), (E), (A1), (C1)$ and (N') implies that

$$G_{\Phi}^{p,\min}(\Delta) \leq G_{\Phi}(\Delta) \leq \tilde{G}_{\Phi}^{p,\max}(\Delta).$$

Proof.

$$\begin{aligned} 0 &\leq \varliminf_{n \rightarrow \infty} \frac{1}{n} \left(G_{\Lambda^n(\mathbf{I} \otimes q^n)}(\Lambda^n(\mathbf{I} \otimes \delta^n)) - G_{\Phi^{\otimes n}}(\Delta^{(n)}) \right) \\ &\leq \varliminf_{n \rightarrow \infty} \frac{1}{n} \left(G_{\mathbf{I} \otimes q^n}(\mathbf{I} \otimes \delta^n) - G_{\Phi^{\otimes n}}(\Delta^{(n)}) \right) \\ &= \varliminf_{n \rightarrow \infty} \frac{1}{n} g_{q^n}(\delta^n) - G_{\Phi}(\Delta) = \varliminf_{n \rightarrow \infty} \frac{1}{n} J_{q^n}(\delta^n) - G_{\Phi}(\Delta). \end{aligned}$$

■

Theorem 12 If $\dim \mathcal{H}_{\text{in}} < \infty$ and $\dim \mathcal{H}_{\text{out}} < \infty$, $G^{p,\min}$ and $G^{p,R}$ satisfy $(M), (E), (A1), (C1)$ and (N') .

Proof. That $G^{p,\min}$ and $G^{p,R}$ satisfy $(M), (E), (A1)$, and (N') is trivial. Hence, we prove $(C1)$. Choose $\rho_{l,\varepsilon}$ so that

$$\frac{1}{l} J_{\Phi^{\otimes l}(\rho_{l,\varepsilon})}^S(\Delta^{(l)}(\rho_{l,\varepsilon})) \geq \frac{1}{l} G_{\Phi^{\otimes l}}^{\min}(\Delta^{(l)}) - \varepsilon.$$

Also, let

$$\begin{aligned} \sigma_{l,\varepsilon} &:= \Phi^{\otimes l}(\rho_{l,\varepsilon}), \\ \delta_{l,\varepsilon} &:= \Delta^{(l)}(\rho_{l,\varepsilon}), \end{aligned}$$

and

$$\begin{aligned} \sigma'_{l,m,\varepsilon} &:= (\Phi^{\otimes m} + \Psi_m)(\rho_{l,\varepsilon}^{\otimes(m/l)}), \\ \delta'_{l,m,\varepsilon} &:= (\Delta^{(m)} + D_m)(\rho_{m,\varepsilon}^{\otimes(m/l)}). \end{aligned}$$

Then

$$\begin{aligned} \left\| \sigma'_{l,m,\varepsilon} - \sigma_{l,\varepsilon}^{\otimes m} \right\|_1 &\leq \|\Psi_m\|_{\text{cb}}, \\ \left\| \delta'_{l,m,\varepsilon} - \delta_{l,\varepsilon}^{(m)} \right\|_1 &\leq \|D_m\|_{\text{cb}}. \end{aligned}$$

Observe by Schwartz's inequality,

$$J_{\sigma_{l,\varepsilon}^{\otimes(m/l)}}^S \left(\delta_{l,\varepsilon}^{(m/l)} \right) \geq \frac{\left| \text{tr} \delta_{l,\varepsilon}^{(m/l)} X \right|^2}{\text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} X^2}$$

and the equality is achieved by

$$X_{m,l,\varepsilon} = \frac{l}{m J_{l,\varepsilon}^S} L_{\sigma_{l,\varepsilon}^{\otimes(m/l)}}^S \left(\delta_{l,\varepsilon}^{(m/l)} \right),$$

where $J_{l,\varepsilon}^S = J_{\sigma_{l,\varepsilon}}^S (\delta_{l,\varepsilon})$. Let $X_{m,l,\varepsilon} = \int x E (dd x)$ be the spectral decomposition, and define $P_a := \int_{x \leq a} E (dd x)$. Then,

$$\begin{aligned} \frac{1}{m} J_{\sigma_{l,m,\varepsilon}}^S (\delta'_{l,m,\varepsilon}) &\geq \frac{\left| \text{tr} \delta'_{l,m,\varepsilon} X_{m,l,\varepsilon} P_a \right|^2}{m \text{tr} \sigma'_{l,m,\varepsilon} (X_{m,l,\varepsilon} P_a)^2} = \frac{\left| \frac{1}{\sqrt{m}} \text{tr} \delta'_{l,m,\varepsilon} X_{m,l,\varepsilon} P_a \right|^2}{\text{tr} \sigma'_{l,m,\varepsilon} (X_{m,l,\varepsilon} P_a)^2} \\ &\geq \frac{\left| \frac{1}{\sqrt{m}} \text{tr} \delta_{l,\varepsilon}^{(m/l)} X_{m,l,\varepsilon} P_a \right|^2}{\text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} (X_{m,l,\varepsilon} P_a)^2} + O(\|\Psi_m\|_{\text{cb}}) + O\left(\frac{1}{\sqrt{m}} \|D_m\|_{\text{cb}}\right) \\ &= \frac{\left| \frac{\sqrt{m}}{l} J_{l,\varepsilon}^S \text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} (X_{m,l,\varepsilon})^2 P_a \right|^2}{\text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} (X_{m,l,\varepsilon} P_a)^2} + O(\|\Psi_m\|_{\text{cb}}) + O\left(\frac{1}{\sqrt{m}} \|D_m\|_{\text{cb}}\right) \\ &= \frac{m}{l^2} (J_{l,\varepsilon}^S)^2 \text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} (X_{m,l,\varepsilon})^2 P_a + O(\|\Psi_m\|_{\text{cb}}) + O\left(\frac{1}{\sqrt{m}} \|D_m\|_{\text{cb}}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left| \frac{m}{l^2} (J_{l,\varepsilon}^S)^2 \text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} (X_{m,l,\varepsilon})^2 - \frac{m}{l^2} (J_{l,\varepsilon}^S)^2 \text{tr} \sigma_{l,\varepsilon}^{\otimes m} (X_{m,l,\varepsilon})^2 P_a \right| \\ &= \left| \frac{m}{l^2} (J_{l,\varepsilon}^S)^2 \text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} (X_{m,l,\varepsilon})^2 (1 - P_a) \right| \\ &\leq \frac{1}{a^2} \left| \frac{m}{l^2} (J_{l,\varepsilon}^S)^2 \text{tr} \sigma_{l,\varepsilon}^{\otimes(m/l)} (X_{m,l,\varepsilon})^4 \right| \\ &= \frac{l^2}{a^2 (J_{l,\varepsilon}^S)^2 m^3} \left\{ \frac{m}{l} \left(\frac{m}{l} - 1 \right) J_{l,\varepsilon}^S + \frac{m}{l} \text{tr} \sigma_{l,\varepsilon} \left(L_{\sigma_{l,\varepsilon}}^S (\delta_{l,\varepsilon}) \right)^4 \right\} \\ &\leq \frac{1}{a^2 (J_{l,\varepsilon}^S)^2 m} \left\{ J_{l,\varepsilon}^S + \text{tr} \sigma_{l,\varepsilon} \left(L_{\sigma_{l,\varepsilon}}^S (\delta_{l,\varepsilon}) \right)^4 \right\} \end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{1}{m} J_{\sigma'_{l,m,\varepsilon}}^S (\delta'_{l,m,\varepsilon}) \\
& \geq \frac{m}{l^2} (J_{l,\varepsilon}^S)^2 \text{tr} \sigma_{l,\varepsilon}^{\otimes m} (X_{m,l,\varepsilon})^2 \\
& + O(\|\Psi_m\|_{\text{cb}}) + O\left(\frac{1}{\sqrt{m}} \|D_m\|_{\text{cb}}\right) - \frac{1}{a^2 (J_{l,\varepsilon}^S)^2 m} \left\{ J_{l,\varepsilon}^S + \text{tr} \sigma_{l,\varepsilon} \left(L_{\sigma_{l,\varepsilon}}^S (\delta_{l,\varepsilon}) \right)^4 \right\} \\
& = \frac{1}{l} J_{l,\varepsilon}^S \\
& + O(\|\Psi_m\|_{\text{cb}}) + O\left(\frac{1}{\sqrt{m}} \|D_m\|_{\text{cb}}\right) - \frac{1}{a^2 (J_{l,\varepsilon}^S)^2 m} \left\{ J_{l,\varepsilon}^S + \text{tr} \sigma_{l,\varepsilon} \left(L_{\sigma_{l,\varepsilon}}^S (\delta_{l,\varepsilon}) \right)^4 \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{1}{m} G_{\Phi^{\otimes m} + \Psi_m}^{p,\min} (\Delta^{(m)} + D_m) \\
& = \lim_{n \rightarrow \infty} \frac{1}{mn} G_{(\Phi^{\otimes m} + \Psi_m)^{\otimes n}}^{\min} \left((\Delta^{(m)} + D_m)^{(n)} \right) \geq \frac{1}{m} G_{\Phi^{\otimes m} + \Psi_m}^{\min} (\Delta^{(m)} + D_m) \\
& \geq \frac{1}{m} J_{\sigma'_{l,m,\varepsilon}}^S (\delta'_{l,m,\varepsilon}) \\
& \geq \frac{1}{l} G_{\Phi^{\otimes l}}^{\min} (\Delta^{(l)}) - \varepsilon \\
& + O(\|\Psi_m\|_{\text{cb}}) + O\left(\frac{1}{\sqrt{m}} \|D_m\|_{\text{cb}}\right) - \frac{1}{a^2 (J_{l,\varepsilon}^S)^2 m} \left\{ J_{l,\varepsilon}^S + \text{tr} \sigma_{l,\varepsilon} \left(L_{\sigma_{l,\varepsilon}}^S (\delta_{l,\varepsilon}) \right)^4 \right\} \\
& \rightarrow \frac{1}{l} G_{\Phi^{\otimes l}}^{\min} (\Delta^{(l)}) - \varepsilon \quad (m \rightarrow \infty) \\
& \rightarrow G_{\Phi}^{p,\min} (\Delta) - \varepsilon \quad (l \rightarrow \infty).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \varliminf_{m \rightarrow \infty} \frac{1}{m} \left\{ G_{\Phi^{\otimes m} + \Psi_m}^{p,\min} (\Delta^{(m)} + D_m) - \frac{1}{m} G_{\Phi^{\otimes m}}^{p,\min} (\Delta^{(m)}) \right\} \\
& \geq G_{\Phi}^{p,\min} (\Delta) - \varepsilon - \varliminf_{m \rightarrow \infty} \frac{1}{m} G_{\Phi^{\otimes m}}^{p,\min} (\Delta^{(m)}) \\
& = G_{\Phi}^{p,\min} (\Delta) - \varepsilon - G_{\Phi}^{p,\min} (\Delta) = -\varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have (C1) for $G^{p,\min}$. (C1) for $G^{p,R}$ is proved almost parallelly, utilizing the following consequence of Schwartz's inequality:

$$J_{\rho}^R (\delta) \geq \frac{|\text{tr} \delta X|^2}{\text{tr} \rho X^{\dagger} X},$$

where X is an arbitrary operator. ■

Corollary 13

$$G_{\Phi}^{p,\min} (\Delta) \leq G_{\Phi}^{p,R} (\Delta) \leq \tilde{G}_{\Phi}^{p,\max} (\Delta).$$

7.3 Quantum states

In [6], it had been essentially proved

$$\begin{aligned} G^{\min} &= G^{\min,p} = G^{\min,a} = J^S, \\ G^{\max} &= G^{\max,p} = G^{\max,a} = J^R. \end{aligned}$$

It is not difficult to see

$$J^R = G^{p,R}.$$

Therefore, we have

$$G^{p,R} = J^R \leq \tilde{G}^{\max,p}.$$

On the other hand, if $\dim \mathcal{H} < \infty$, using [8], $J^R \geq \tilde{G}^{\max,p}$. Thus,

$$G^{\max} = G^{\max,p} = G^{\max,a} = \tilde{G}^{\max,p} = J^R.$$

The following theorem is a corollary of Theorem 12.

Theorem 14 J_ρ^S and J_ρ^R satisfies (C), i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(J_{\rho'^n}^S(\delta'^n) - J_{\rho^{\otimes n}}^S(\delta^{(n)}) \right) \geq 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \left(J_{\rho'^n}^R(\delta'^n) - J_{\rho^{\otimes n}}^R(\delta^{(n)}) \right) \geq 0$$

if $\|\rho'^n - \rho^{\otimes n}\|_1 \rightarrow 0$ and $\frac{1}{\sqrt{n}} \|\delta'^n - \delta^{(n)}\|_1 \rightarrow 0$.

7.4 Classical channel

By [8], we have the following :

Theorem 15 Suppose p is a probability distribution and δ is a signed measure over set with k -elements ($k < \infty$). Let $J := J_p(\delta)$, $\varepsilon > 0$ and

$$\{q^n, \delta'^n\} := \left\{ \mathcal{N}(0, 1), \sqrt{n(J + \varepsilon)} \delta \mathcal{N}(0, 1) \right\} \equiv \left\{ \mathcal{N}(0, 1), \delta \mathcal{N}(0, 1) \right\}^{\otimes n(J + \varepsilon)}.$$

Then, we can compose an asymptotic parallel tangent simulation of $\{p^{\otimes n}, \delta^{(n)}\}$ using $\{q^n, \delta'^n\}$.

Theorem 16 $\Phi(\cdot|x)$ is a probability distribution and $\Delta(\cdot|x)$ is a signed measure over set with k -elements ($k < \infty$). Let us define $\{q_\varepsilon^n, \delta_\varepsilon^n\} := \left\{ \mathcal{N}(0, 1), \delta \mathcal{N}(0, 1) \right\}^{\otimes n(1 + k\varepsilon)J}$ where $J = G_\Phi^{\min}(\Delta) = \max_{1 \leq x \leq k} J_{\Phi(\cdot|x)}(\Delta(\cdot|x))$ and ε is arbitrary positive number. Then, there is Λ^n such that

$$\begin{aligned} \left\| \Phi^{\otimes n}(p) - \Lambda^n(p \otimes q_\varepsilon^n) \right\|_{\text{cb}} &\leq \frac{k}{\sqrt{\varepsilon n}} \max_x C(\{\Phi(\cdot|x), \Delta(\cdot|x)\}), \\ \frac{1}{\sqrt{n}} \left\| \Delta^{(n)}(p) - \Lambda^n(p \otimes \delta_\varepsilon^n) \right\|_{\text{cb}} &\leq \frac{k}{\sqrt{\varepsilon n}} \max_x C(\{\Phi(\cdot|x), \Delta(\cdot|x)\}). \end{aligned}$$

7.5 CQ channel

Let

$$\Phi(x) = \rho_x, \quad \Delta(x) = \delta_x.$$

Then, trivially

$$G_{\Phi}^{\min,p}(\Delta) = \max_x J_{\rho_x}^S(\delta_x).$$

Also,

$$\tilde{G}_{\Phi}^{\max,p}(\Delta) \geq G_{\Phi}^{R,p}(\Delta) = \max_x J_{\rho_x}^R(\delta_x).$$

In the sequel, we prove $\tilde{G}_{\Phi}^{\max,p}(\Delta) \leq G_{\Phi}^{R,p}(\Delta) = \max_x J_{\rho_x}^R(\delta_x)$, if $\dim \mathcal{H} < \infty$. Let $\{q_x, \delta'_x\}$ be the optimal tangent simulation of $\{\rho_x, \delta_x\}$, or satisfy

$$\rho_x = \sum_y q_x(y) \rho_{x,y}, \quad \delta_x = \sum_y \delta'_x(y) \rho_{x,y},$$

and $J_{q_x}(\delta'_x) = J_{\rho_x}^R(\delta_x)$ (see [6]).

Denote $\max_x J_{\rho_x}^R(\delta_x)$ by J . Given a input sequence $x^n = x_1 x_2 \cdots x_n$, denote by n_x the times of $x_i = x$ in the sequence x^n . Suppose $n_x \geq \varepsilon n$. Then, we use $\{N(0, 1), \delta N(0, 1)\}^{\otimes n_x J}$ for simulation of $\{\rho_x^{\otimes n_x}, \delta_x^{(n_x)}\}$. On the other hand, if $n_x < \varepsilon n$, we first fabricate $\{\rho_x^{\otimes \varepsilon n}, \delta_x^{(\varepsilon n)}\}$ consuming $\{N(0, 1), \delta N(0, 1)\}^{\otimes n_x J}$, and takes partial trace. In both case, by Theorem 15, the error of simulation vanishes as $n \rightarrow \infty$. We do this for all $x = 1, \dots, k$. As a whole, we used $\{N(0, 1), \delta N(0, 1)\}^{\otimes (n + \varepsilon k n) J}$ to simulate $\bigotimes_{\kappa=1}^n \{\rho_{x_{\kappa}}^{\otimes n_{x_{\kappa}}}, \delta_{x_{\kappa}}^{(n_{x_{\kappa}})}\}$. Since $\varepsilon > 0$ is arbitrary, we have

$$\tilde{G}_{\Phi}^{\max,p}(\Delta) = G_{\Phi}^{R,p}(\Delta) = \max_x J_{\rho_x}^R(\delta_x).$$

Conjecture 17 For a QC channel, $G_{\Phi}^{R,p}(\Delta) = \tilde{G}_{\Phi}^{\max,p}(\Delta) \not\leq G_{\Phi}^{\max,p}(\Delta)$.

References

- [1] S. Amari, Differential-geometrical methods in statistics, Lecture Notes in Statistics, 28 (1985).
- [2] S. Amari and H. Nagaoka, Methods of Information Geometry, Translations of Mathematical Monograph, Vol. 191 (AM Sand Oxford University Press, 2000).
- [3] Akio Fujiwara and Hiroshi Imai, "Quantum parameter estimation of a generalized Pauli channel," J. Phys. A: Math. Gen., vol. 36, pp. 8093-8103 (2003).

- [4] N. N. Cencov: Statistical Decision Rules and Optimal Inference. Trans. of Mathematical Monographs 53, Amer. Math. Soc., Providence (1982).
- [5] K. Matsumoto, A Geometrical Approach to Quantum Estimation Theory, doctoral dissertation, University of Tokyo, 1998.
- [6] K. Matsumoto, Reverse estimation theory, Complementarity between RLD and SLD, and monotone distances, arXiv:quant-ph/0511170 (2005).
- [7] K. Matsumoto, On metric of classical channel spaces: non-asymptotic theory
- [8] K. Matsumoto, On metric of classical channel spaces: asymptotic theory
- [9] K. Matsumoto, On the First Order Asymptotic Theory of Quantum Estimation,
- [10] H. Nagaoka, "On the Parameter Estimation Problem for Quantum Statistical Models," SITA'89, 577-582 Dec. (1989).
- [11] D. Petz, Monotone metrics on matrix spaces, Linear Algebra Appl., 244,81-96 (1996).
- [12] E. Torgersen, "Comparison of statistical experiments," (Cambridge university press ,1991).